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# SCALING EXPONENTS AND FLUCTUATION STRENGTH IN HIGH ENERGY COLLISIONS

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## ABSTRACT

The information on dynamical fluctuations that can be extracted from the anomalous scaling observed recently in hadron-hadron collision experiments is discussed in some detail. A parameter “effective fluctuation strength” is proposed to estimate the strength of dynamical fluctuations. The method for extracting its value from the experimentally observed quantities is given. Some examples for the application of this method to real experimental data are presented.

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The JACEE experiment in 1983<sup>[1]</sup> and afterwards some accelerator experiments<sup>[2,3]</sup> indicated that there are dynamical fluctuations beyond the usual statistical ones in high energy multiparticle final states. These experimental phenomena had caused high interest in studying the anomalous scaling in high energy collisions<sup>[4]</sup>.

Recently, the expected anomalous scaling has been observed successfully in both the data of  $\pi^+p$  and  $K^+p$  collisions at 250 GeV/c from NA22<sup>[5]</sup> and the data of pp collisions at 400 GeV/c<sup>[6]</sup> from NA27, under the assumption that the dynamical fluctuations are anisotropic in longitudinal-transverse planes<sup>[7]</sup>.

It is now natural to ask the question: What is the underlying physics of these experimental findings, and/or what are the properties of the dynamical fluctuations that lead to the experimentally observed anomalous scaling. This question is the subject of a long term study. In this letter we want to try a limited task in this direction, i.e. to study the relation between the observed anomalous scaling exponents and the strength of dynamical fluctuations within the framework of a simple model — the random cascading  $\alpha$ -model.

Firstly, let us recall that what is observed directly in experiments is the anomalous scaling of normalized factorial moments  $F_q$ , which under the assumption of Poisson or Bernoulli type of statistical fluctuations are equal to the normalized probability moments<sup>[8]</sup>:

$$F_q = \frac{1}{M} \sum_{m=1}^M \frac{\langle p_m^q \rangle}{\langle p_m \rangle^q}. \quad (1)$$

Here, a phase space region  $\Delta$  is divided into  $M$  sub-cells<sup>1</sup> and  $p_m$  is the probability for finding particles in the  $m$ th sub-cell. The anomalous scaling of  $F_q$ :

$$F_q(M) \propto M^{\phi_q} \quad (M \rightarrow \infty) \quad (2)$$

with non-vanishing indices  $\phi_q$ , called intermittency (IM) indices, is an evidence for the existence of dynamical fluctuations<sup>[8]</sup>.

On the other hand, in studying the fractal property of the system, one usually uses the un-normalized moments<sup>[9]</sup>

$$C_q = \sum_{m=1}^M \langle p_m^q \rangle, \quad (3)$$

which has the anomalous scaling property

$$C_q(M) \propto M^{-(q-1)D_q} \quad (M \rightarrow \infty). \quad (4)$$

The indices  $D_q$ , called multifractal Rényi dimensions<sup>[9]</sup>, are related to the IM indices  $\phi_q$  as

$$D_q = 1 - \frac{\phi_q}{q-1}. \quad (5)$$

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<sup>1</sup>Throughout this paper we use the convention popular in the study of intermittency and fractality in high energy physics, i.e. let  $M$  in Eq's. (1) – (4) be the number of sub-cells in the division of the phase space region  $\Delta$  of topological dimension  $D_T$ . This means, in particular, that for a normal geometrical object the fractal dimension defined via Eq.(4) is always equal to unity. On the contrary, in the usual definition of fractal dimension<sup>[9]</sup>:  $C_q(l) \propto l^{(q-1)D_q}$  ( $l \rightarrow 0$ ),  $l$  is the 'edge' (1-dimensional scale) of the hypercube (sub-cell), and consequently for a normal geometrical object  $D_q = D_T$ .

The question in front of us is as the following: When we obtain successfully a strict scaling of normalized factorial moments through a proper way of phase space division, as for example in the cases of Ref.'s [5] and [6], we are ready to get the slopes of the  $\ln F_q \sim \ln M$  plots, which gives the IM indices and Rényi dimensions. Then, how to extract a characteristic quantity describing the strength of dynamical fluctuations from the values of these indices and dimensions. This is the problem we are about to discuss.

It should be noticed that an anomalous power law and a fractal property of a system both result from the dynamical fluctuations in this system, and therefore the IM indices  $\phi_q$  and multifractal Rényi dimensions  $D_q$  are related to the strength of dynamical fluctuations. They have, however, their own physical meaning and cannot be taken directly as a measure of the dynamical-fluctuation strength. For example<sup>[9]</sup>, the first-order Rényi dimension  $D_1$ , or equivalently the information dimension  $D_I$ , determines the scaling property of the number of boxes containing the dominant part of information; the second-order Rényi dimension  $D_2$ , sometimes called correlation dimension  $\mu$ , measures the scaling properties of two particle correlations, etc. These scaling properties come from a common origin — the dynamical fluctuations. All of them are related to these fluctuations but none of them can serve as an appropriate quantity for describing the strength of these fluctuations directly.

In order to show the relation between the strength of dynamical fluctuations and the values of IM indices or Rényi dimensions, let us consider a simple example — the random cascading  $\alpha$ -model<sup>[8,10]</sup>. This model describes each multiparticle event as a series of steps, in which the initial phase space region  $\Delta$  is repeatedly divided into  $\lambda = 2$  parts. After  $\nu$  steps we get  $M = 2^\nu$  sub-cells of size  $\delta = \Delta/M$ . At each step  $\nu$  the normalized particle density is obtained in each of the two parts by multiplication of the normalized density in the step  $\nu - 1$  by a particular value of the random variable  $\omega_{\nu j_\nu}$ , where  $j_\nu$  is the position of a sub-cell at the  $\nu$ th step ( $1 \leq j_\nu \leq 2^\nu$ ).

The elementary fluctuation probability  $\omega$  can be chosen in various ways<sup>[8,10]</sup>. The simplest way that provides a characteristic parameter for describing the fluctuation strength is to choose it as<sup>[10]</sup>

$$\omega_{\nu, 2j-1} = \frac{1}{2}(1 + \alpha r) \quad ; \quad \omega_{\nu, 2j} = \frac{1}{2}(1 - \alpha r), \quad (6)$$

in which  $r$  is a random number distributed uniformly in the interval  $[-1, 1]$ ,  $j$  is an integer ( $1 \leq j \leq 2^{\nu-1}$ ),  $\alpha$  is a positive constant taking value in the range  $[0, 1]$ ,

$$0 \leq \alpha \leq 1. \quad (7)$$

The value of  $\alpha$  determines the possible region of  $\omega$ ,

$$\frac{1 - \alpha}{2} < \omega < \frac{1 + \alpha}{2}.$$

Let us note that  $\omega$  defines the way how particles are distributed from step to step between the two pieces of a given cell, i.e. it characterizes the strength of multiplicity

fluctuations in cell-division, and  $\alpha$  determines the width of the possible values of  $\omega$ , therefore,  $\alpha$  is the characteristic quantity describing the *strength of dynamical fluctuations* in this version of random cascading model. In the following, we will analyse the relation between the strength parameter  $\alpha$  and the multifractal dimensions  $D_q$  in this model.

In the random cascading  $\alpha$ -model

$$F_q(M) = \frac{\langle \omega^q(1) \cdots \omega^q(\nu) \rangle}{\langle \omega \rangle^{q\nu}}. \quad (8)$$

In the limit of large  $\nu$ , the distribution of random variable  $\zeta = \sum_{i=1}^{\nu} \ln \omega(i)$  approaches a Gaussian<sup>[8]</sup>

$$p(\zeta)d\zeta = (2\pi\nu)^{-1/2} \sigma^{-1} \exp[-(\zeta - \nu\bar{V})^2/2\nu\sigma^2]d\zeta$$

with

$$\sigma^2 = \int P(\omega)(\ln(\omega) - \bar{V})^2 d\omega, \quad \bar{V} = \int P(\omega) \ln(\omega) d\omega. \quad (9)$$

Explicit calculation of multifractal dimensions in this limit gives

$$D_q = 1 - \frac{1}{2 \ln \lambda} \sigma^2 q. \quad (10)$$

A characteristic feature of the Gaussian approximation is the proportionality of  $1 - D_q$  and  $q$  as can be seen from Eq.(10):

$$\frac{1 - D_q}{q} = \frac{1}{2 \ln 2} \sigma^2.$$

Here and in the following we take  $\lambda = 2$  for simplicity. In Fig.1 is shown the relation between  $1 - D_q$  and  $q$  for  $\alpha = 0.1$  to  $0.5$ . It can be seen from the figure that when  $\alpha$  is small there is a fairly good linear relation between  $1 - D_q$  and  $q$ , so that the Gaussian approximation is sufficiently good in these cases.

To get a relation between  $D_q$  and  $\alpha$ , we calculate the variance  $\sigma^2$ , appearing in eq.(10), of the random variable  $\ln \omega$

$$\sigma^2 = \langle \ln^2 \omega \rangle - \langle \ln \omega \rangle^2 = \frac{1}{3} \alpha^2 + \frac{2}{3} \alpha^4 + \cdots. \quad (11)$$

Under linear approximation

$$\sigma^2 \approx \frac{1}{3} \alpha^2. \quad (12)$$

How  $\sigma$  is related to  $\alpha$  is shown in Fig.2. In this figure, the full circles represent the result without linear approximation, the dashed line indicates the linear-approximation. When  $\alpha$  is not very large ( $\alpha \leq 0.5$  say), the two results are nearly equal. Therefore, both Gaussian and linear approximations can be used when  $\alpha \leq 0.5$ . This region of  $\alpha$  is sufficient for the limited range available in actual experiments, cf. Table I.

Substituting Eq.(12) into Eq.(10), we get

$$D_q = 1 - \frac{1}{6 \ln 2} q \alpha^2$$

or

$$\alpha = \sqrt{\frac{6 \ln 2}{q} (1 - D_q)}. \quad (13)$$

It can be seen from Eq.(13) that, in the random cascading  $\alpha$ -model, with small  $\alpha$ , the strength parameter  $\alpha$  of dynamical fluctuations is related to the multifractal dimensions  $D_q$  by a very simple relation. Using Eq.(13) we can get an approximate value of the fluctuation parameter  $\alpha$  as long as the multifractal dimension of any order  $q$  is known. In high energy experiments, the second-order multifractal dimension  $D_2$  is most easy to obtain. For this reason, the r.h.s. of Eq.(13) for  $q = 2$ ,  $\sqrt{3 \ln 2 (1 - D_2)} \approx \sqrt{2(1 - D_2)}$ , can be taken as a characteristic quantity for the strength of dynamical fluctuations.

In Fig.3 the values of  $\sqrt{2(1 - D_2)}$ , denoted by  $\alpha_{\text{eff}}$  (cf. Eq.(14)), are plotted against the model parameter  $\alpha$  varying from 0 to 1. The dashed line corresponds to  $\alpha_{\text{eff}} = \alpha$ . From the figure we can see that  $\sqrt{2(1 - D_2)}$  has almost equal value with  $\alpha$ , especially when  $\alpha$  is not very large. This means that  $\sqrt{2(1 - D_2)}$  represents the value of  $\alpha$  fairly well. Thus, we have been successful in obtaining an estimation of the fluctuation strength  $\alpha$  in terms of the second-order fractal dimension  $D_2$  in the framework of the random cascading  $\alpha$ -model.

The above results are obtained from a special model. In the general case, when the underlying dynamics is unclear, it is hard to define the “strength of dynamical fluctuations” strictly. In these cases, we can make use of the results obtained from the random cascading  $\alpha$ -model as an estimation for this strength.

Thus, for an arbitrary process that has anomalous scaling property, we define an *effective fluctuation strength*

$$\alpha_{\text{eff}} = \sqrt{2(1 - D_2)} = \sqrt{2\phi_2}. \quad (14)$$

as an estimation of the strength of the dynamical fluctuations taking place in this process. Its physical meaning is:

*The effective fluctuation strength  $\alpha_{\text{eff}}$  of an arbitrary process is the fluctuation strength of a random cascading  $\alpha$ -model with elementary partition number  $\lambda = 2$  that can give the same value of second-order IM index  $\phi_2$  (within the Gaussian and linear approximation) as this process.*

In getting the last equality of Eq.(14) the relation (5) between the IM indices and Rényi dimensions has been used.

Using the effective fluctuation strength defined above, we are now able to compare the strength of dynamical fluctuations in different collision processes. In the following we will give some examples. Before doing that, a question has still to be considered.

In real experiments the dynamical fluctuations exist in higher-dimension<sup>[11]</sup> and are usually anisotropic<sup>[5–7]</sup> with a particular value of Hurst exponent  $H_{\parallel\perp}$ . How can we use the definition (14) of effective fluctuation strength, which depends on a one-dimensional  $\alpha$ -model with elementary partition number  $\lambda = 2$ , to these cases?

In answering this question, let us note that in case of anisotropic dynamical fluctuation (self-affine fractal), the partition numbers in different phase space directions cannot be simultaneously equal to integer values. The method of factorial moment analysis with non-integer partition<sup>[12,13]</sup> has to be used and the resulting  $F_2^{3D}$  for arbitrary value of  $M^{3D}$  all lie on a same straight line. Therefore, we can freely choose  $M^{3D} = 2^\nu$ ,  $\nu = 1, 2, \dots$ . The anomalous scaling property of such a 3-D fractal is equivalent to that of a 1-D fractal with elementary partition number  $\lambda = 2$ . Therefore, our definition (14) for effective fluctuation strength is applicable also to this case.

Now, let us turn to the examples for the application of effective fluctuation strength to real experimental data. For this purpose we have to choose those data that possess good anomalous scaling property.

For hadron-hadron collisions, the presently available 3-D data that have good scaling property are the self-affinely analysed data for  $\pi^+p$  and  $K^+p$  collisions at 250 GeV/c from NA22<sup>[5]</sup> with Hurst exponents  $H_{yp_t} = H_{y\varphi} = 0.475$ ,  $H_{p_t\varphi} = 1$ . The second-order IM index is obtained as<sup>[13]</sup>:  $\phi_2^{3D} = 0.061 \pm 0.010$ .

Another example of hadron-hadron collisions is the pp collisions at 400 GeV/c from NA27<sup>[6]</sup>. In this case, only 2-D  $(\eta, \varphi)$  data are available due to lack of momentum measurement. The Hurst exponent in the  $(\eta, \varphi)$  plane is found to be  $H_{\eta\varphi} = 0.74$ . Fitting the results to a straight line gives  $\phi_2^{2D} = 0.051 \pm 0.004$ .

As an example of  $e^+e^-$  collisions we take the data from DELPHI<sup>[14]</sup>. After omitting the first point to eliminate the influence of momentum conservation<sup>[15]</sup>, a good fit to a straight line comes out<sup>2</sup>, cf. Fig.4. The second-order IM index is then obtained as  $\phi_2^{3D} = 0.099 \pm 0.005$ .

The resulting effective fluctuation strengths  $\alpha_{\text{eff}}$  for these three cases are listed in the last column of Table I. Its physical meaning is that, the anomalous scaling of the second-order factorial moments in these 3 experiments can be produced by random cascading processes having fluctuation strength  $\alpha \approx 0.349, 0.319, 0.446$ , respectively.

Table I The Hurst exponents, IM indices, second order Rényi dimensions and effective fluctuation strengths for the data from 3 experiments

Experiment	Hurst exponent	$\phi_2$	$D_2$	$\alpha_{\text{eff}}$
NA22 (3D)	$H_{y\perp} = 0.475, H_{p_t\varphi} = 1$	$0.061 \pm 0.010$	$0.939 \pm 0.010$	$0.349 \pm 0.028$
NA27 (2D)	$H_{\eta\varphi} = 0.74$	$0.051 \pm 0.004$	$0.949 \pm 0.004$	$0.319 \pm 0.014$
DELPHI (3D)	$H = 1$	$0.099 \pm 0.005$	$0.901 \pm 0.005$	$0.446 \pm 0.012$

Let us notice that all the values of  $\phi_2$  and  $D_2$  listed in Table I are near by the boundaries of their allowed range, i.e. 0 for  $\phi_2$  and 1 for  $D_2$ . This would give us

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<sup>2</sup>The reason why the  $e^+e^-$  data have good scaling property already for the Hurst exponent  $H = 1$  will be discussed elsewhere.

an impression that the strength of dynamical fluctuations in these experiments are all marginal. This is, however, wrong. From the last column of Table I we can see that the values of  $\alpha_{\text{eff}}$  for hadron-hadron collisions are approximately equal to  $1/3$  while that for  $e^+e^-$  collisions is close to  $1/2$ . Since the allowed range of the parameter  $\alpha$  characterizing dynamical fluctuations is  $[0, 1]$ , cf. Eq.(7), the above results show that the dynamical fluctuations are nearly equal to one third of the maximum possible strength in hadron-hadron collisions and about half of the maximum possible strength in  $e^+e^-$  case. This gives us, at least qualitatively, a feeling about the strength of dynamical fluctuations in these collision processes.

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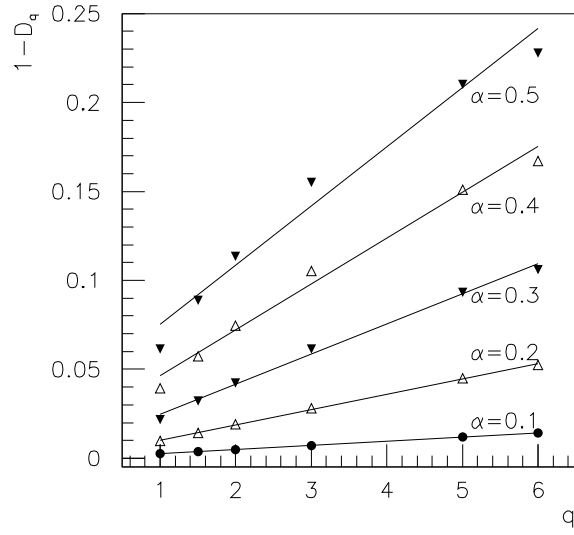


Fig. 1 Relation between the  $1 - D_q$  and  $q$  for 5 values of model parameter  $\alpha$ .

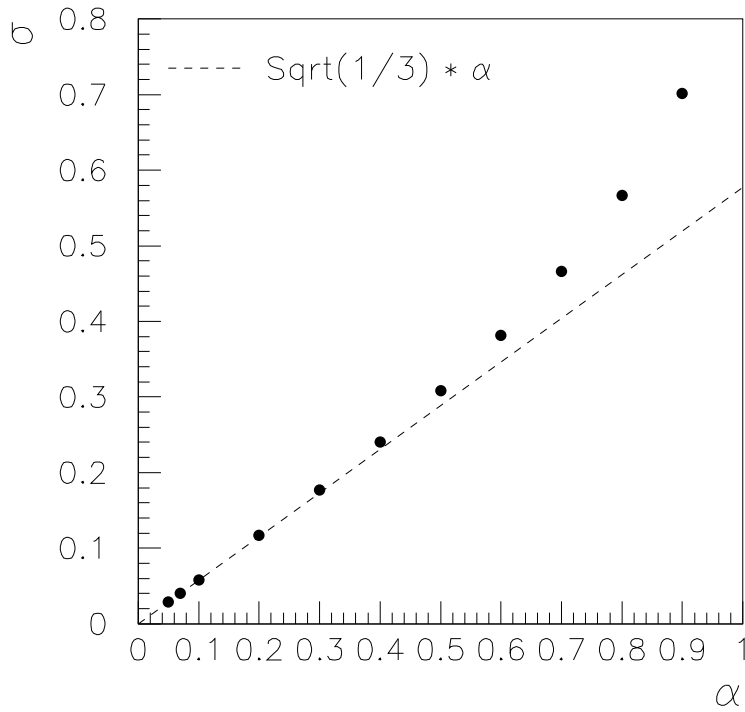


Fig. 2 Relation between the standard deviation  $\sigma$  of random variable  $\ln \omega$  and model parameter  $\alpha$ .



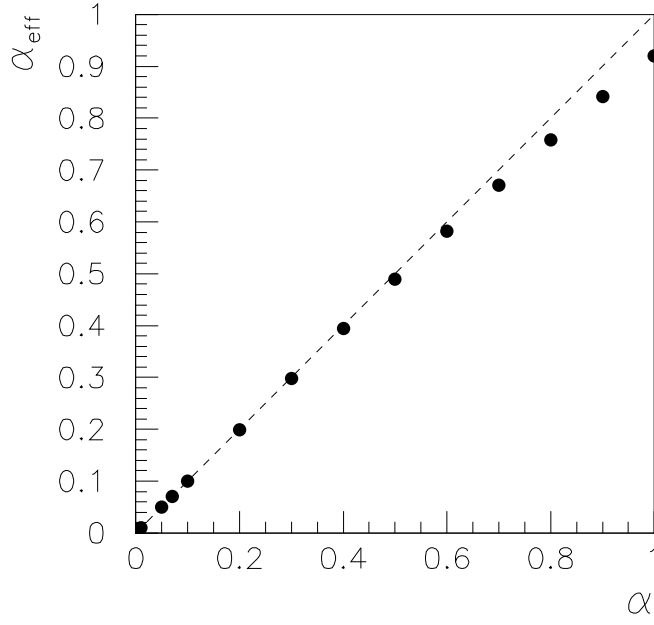


Fig. 3 Relation between the effective fluctuation strength  $\alpha_{\text{eff}}$  and model parameter  $\alpha$ .  
The dashed line corresponds to  $\alpha_{\text{eff}} = \alpha$

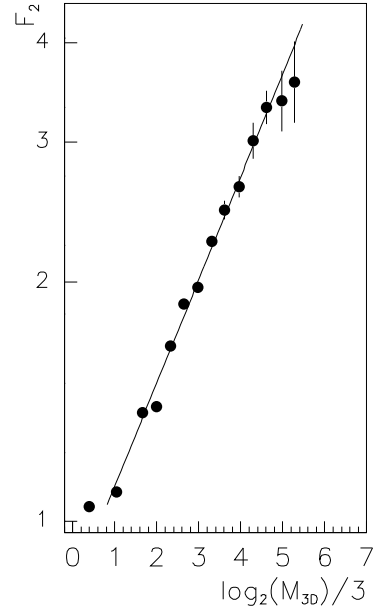


Fig. 4 The anomalous scaling of second order 3-D factorial moment of  $e^+e^-$  collisions at 91 GeV (data taken from Ref.[14]).